Variables Scaling to Solve a Singular Bifurcation Problem with Applications to Periodically Perturbed Autonomous Systems

(Dedicated to Prof. R. Johnson on the occasion of his 60th birthday)

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Abstract. By means of a linear scaling of the variables we convert a singular bifurcation equation in $\mathbb{R}^n$ into an equivalent equation to which the classical implicit function theorem can be directly applied. This allows to deduce the existence of a unique branch of solutions as well as a relevant property of the spectrum of the derivative of the singular bifurcation equation along the branch. We use these results to show the existence, uniqueness and the asymptotic stability of periodic solutions of a T-periodically perturbed autonomous system bifurcating from a T-periodic limit cycle of the autonomous unperturbed system. This problem is classical, but the novelty of the method proposed is that it allows us to solve the problem without any reduction of the dimension of the state space as it is usually done in the literature by means of the Lyapunov-Schmidt method.

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1 Introduction

In Section 2 we consider an abstract bifurcation equation of the form

$$\Phi(v,\varepsilon) := P(v) + \varepsilon Q(v,\varepsilon) = 0 \tag{1.1}$$

where $P \in C^2(\mathbb{R}^n, \mathbb{R}^n), Q \in C^1(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$ and, for $\varepsilon > 0$ sufficiently small, we look for the existence of zeros v_ε of the map Φ. Here it is assumed the existence of a $v_0 \in \mathbb{R}^n$ such that $P(v_0) = 0$ with the matrix $P'(v_0)$ singular. In other words, we deal with an abstract singular bifurcation problem in $\mathbb{R}^n$ with a small bifurcation parameter $\varepsilon > 0$. Due to the singularity of $P'(v_0)$ it is not possible to use directly to (1.1) the classical implicit function theorem to show the existence and uniqueness of a branch $\{v_\varepsilon\}$, $\varepsilon > 0$ small, of solutions of the equation $\Phi(v,\varepsilon) = 0$. In this paper, by means of a linear scaling of the variables $v \in \mathbb{R}^n$ we convert the problem of finding zeros of (1.1) to the problem of finding zeros of a map $\Psi(w,\varepsilon)$ for which there exists a unique $w_0 \in \mathbb{R}^n$ such that $\Psi(w_0, 0) = 0$ and $\Psi'_w(w_0, 0)$ is not singular. Therefore, the new bifurcation equation $\Psi(w,\varepsilon) = 0$ can be solved by means of the classical implicit function theorem to conclude the existence and uniqueness of a branch of zeros $\{w_\varepsilon\}$, for $\varepsilon > 0$ small. The advantage and the novelty of the approach is that getting the equation $\Psi(w,\varepsilon) = 0$ does not require solving any implicit equations which is usually done when applying the Lyapunov-Schmidt reduction approach (see [3], Ch. 2, § 4).

Our bifurcation equation $\Psi(w,\varepsilon) = 0$ is, therefore, formally different from that given by Lyapunov-Schmidt reduction (see e.g. [9]). That is why we show in Section 3 that applying our general result to the perturbed autonomous system

$$\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon). \tag{1.2}$$

where $f \in C^2(\mathbb{R}^n, \mathbb{R}^n), g \in C^1(\mathbb{R} \times \mathbb{R}^n \times [0,1], \mathbb{R}^n)$ is T-periodic and $\varepsilon > 0$ is small, leads to the same classical Malkin-Loud (or sometimes called Melnikov) bifurcation function. We end up, therefore, with the statement that a well known classical result on the existence, uniqueness and asymptotic stability of a family of T-periodic solution of (1.2) bifurcating from the T-periodic limit cycle x_0 of the autonomous system $\dot{x} = f(x)$ (see Malkin [11], Loud [9], Blekhman [1]) follows from our bifurcation theorem, while avoiding the Lyapunov-Schmidt reduction reduces the analysis significantly.

A first result in this direction has been obtained by the authors in [6] by means of a version of the implicit function theorem for directionally continuous functions, see [2]. The idea of using the linear scaling has been, therefore, already reported at the conference [6]. But the approach in [6] is based on the employ of isochronous surfaces of the Poincaré map transversally intersecting the

limit cycle x_0 that requires a non-trivial information about smoothness of these surfaces, while the considerations in this paper rely on very basic facts of analysis only.

The paper is organized as follows. In Section 2 we first reduce the abstract singular bifurcation equation (1.1) to an equivalent non-singular bifurcation equation, then in Theorem 1 we provide conditions under which the non-singular problem satisfies the assumptions of the classical implicit function theorem. Furthermore, in Theorem 2 we establish a relevant property of the spectrum of the derivative of the singular bifurcation equation along the branch which permits to study the asymptotic stability of the bifurcating zeros. In Section 3, under the standard assumption that the Malkin's bifurcation function associated to (1.2) has non-degenerate zeros, the results stated in Section 2 permit to show (Theorem 3) the existence of a parametrized family of T-periodic solutions of (1.2) bifurcating from the T-periodic limit cycle of the unperturbed system as well as their asymptotic stability. The main tools to prove Theorem 3 consist in a representation formula for the Malkin's bifurcation function in terms of the T-periodic perturbation of the autonomous system and of a formula for its derivative. These formulas are stated in Lemma 2 and Lemma 3 respectively.

2 Variables scaling to transform a singular bifurcation problem into a non-singular one

Consider the function $\Phi : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ defined by

$$\Phi(v,\varepsilon) = P(v) + \varepsilon Q(v,\varepsilon) \tag{2.1}$$

where $P \in C^2(\mathbb{R}^n,\mathbb{R}^n), Q \in C^1(\mathbb{R}^n \times [0,1],\mathbb{R}^n)$ and $\varepsilon > 0$ is a small parameter.
In this Section, assuming the existence of $v_0 \in \mathbb{R}^n$ such that $P(v_0) = 0$ with $P'(v_0)$ singular, we provide a method to show the existence and the uniqueness of the solution v_ε of the equation

$$\Phi(v,\varepsilon) = 0$$

for $\varepsilon > 0$ sufficiently small, without using the usual Lyapunov-Schmidt reduction approach. To this aim we assume the existence of a linear projector $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathrm{Im}\,\Pi \bigoplus \mathrm{Ker}\,\Pi = \mathbb{R}^n$, $\mathrm{Im}\,\Pi$ and $\mathrm{Ker}\,\Pi$ are invariant subspaces under $P'(v_0)$ and $\Pi P'(v_0) = \Pi Q(v_0, 0) = 0$.
Since $P'(v_0)$ is singular we cannot apply the classical implicit function theorem, see e.g. [8], to study the existence of connected components of zeros of Φ emanating from $(v_0, 0)$. Observe that, in general, as it is shown in [9] and [10], there could exist several branches of zeros of Φ emanating from $(v_0, 0)$. In this paper we provide conditions (which are apparently generic when applying the result to differential equations, see Section 3) under which the branch is unique. In particular in

Section 3, such conditions are expressed in terms of the Malkin bifurcation function associated to (1.2), see [11]. More precisely, in Section 3 we have $v_0 = x_0(\theta_0)$, where x_0 is a one parameter curve of zeros of P and θ_0 is a non-degenerate simple zero of the Malkin bifurcation function. The approach to achieve this result is commonly based on the classical Lyapunov-Schmidt reduction method. In the infinite dimensional case, see [5] and more recently [7].

In this paper we propose a different approach based on an equivalent formulation of the problem. More precisely, by means of a scaling of the variables, we rewrite the problem of finding zeros of $\Phi(v,\varepsilon)$, for $\varepsilon > 0$ small, as a non-singular bifurcation problem to which apply the classical implicit function theorem. Namely, we associate to the map Φ the following function

$$\Psi(w,\varepsilon) = \frac{1}{\varepsilon}\left(\Phi(v_0+\varepsilon w,\varepsilon) - \Pi\Phi(v_0+\varepsilon w,\varepsilon) + \frac{1}{\varepsilon}\Pi\Phi(v_0+\varepsilon w,\varepsilon)\right), \tag{2.2}$$

for any $w \in \mathbb{R}^n$ and any $\varepsilon > 0$, and we look for zeros of Ψ branching from some $(w_0, 0)$. Indeed, as it is easy to see, $(v,\varepsilon) \in \mathbb{R}^n \times [0,1]$ is a zero of Φ if and only if $\left(\dfrac{v-v_0}{\varepsilon},\varepsilon\right)$ is a zero of Ψ.

In the sequel the vector space of linear operators $L : \mathbb{R}^n \to \mathbb{R}^n$ will be denoted by $\mathcal{L}(\mathbb{R}^n)$. Next Lemma provides the main properties of the function Ψ.

Lemma 1 *Assume that $P \in C^2(\mathbb{R}^n,\mathbb{R}^n)$ and $Q \in C^1(\mathbb{R}^n \times [0,1],\mathbb{R}^n)$. Let $v_0 \in \mathbb{R}^n$ be such that $P(v_0) = 0$ and $P'(v_0)$ singular. Let $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ be a linear projector invariant with respect to $P'(v_0)$ such that $\Pi P'(v_0) = \Pi Q(v_0,0) = 0$. Define $\Psi(w,0)$ as follows*

$$\Psi(w,0) = \frac{1}{2}\Pi P''(v_0)ww + \Pi Q'_v(v_0,0)w + \Pi Q'_\varepsilon(v_0,0) + (I-\Pi)P'(v_0)w + (I-\Pi)Q(v_0,0) \tag{2.3}$$

with

$$\Psi'_w(w,0) = \Pi P''(v_0)w + \Pi Q'_v(v_0,0) + (I-\Pi)P'(v_0). \tag{2.4}$$

Then $\Psi \in C^0(\mathbb{R}^n \times \mathbb{R},\mathbb{R}^n)$ and $\Psi'_w \in C^0(\mathbb{R}^n \times \mathbb{R},\mathcal{L}(\mathbb{R}^n))$.

Proof. From (2.2) the Taylor expansion with the rest in the Lagrange's form leads to

$$\begin{aligned}\Pi\Psi(w,\varepsilon) &= \frac{1}{\varepsilon^2}\Pi\Phi(v_0+\varepsilon w,\varepsilon) = \frac{1}{\varepsilon^2}\Pi(P(v_0+\varepsilon w) + \varepsilon Q(v_0+\varepsilon w,\varepsilon)) = \\ &= \frac{1}{\varepsilon^2}\Pi\left(P(v_0) + \varepsilon P'(v_0)w + \frac{1}{2}\varepsilon^2 P''\left(v_0+\widehat{\varepsilon}(w,\varepsilon)w\right)ww + \varepsilon Q(v_0,0) + \right. \\ &\quad \left. + \varepsilon^2 Q'_v\left(v_0+\widetilde{\varepsilon}(w,\varepsilon)w,\widetilde{\varepsilon}(w,\varepsilon)\right)w + \varepsilon^2 Q'_\varepsilon\left(v_0+\widetilde{\varepsilon}(w,\varepsilon)w,\widetilde{\varepsilon}(w,\varepsilon)\right)\right)\end{aligned}$$

and

$$\begin{aligned}(I-\Pi)\Psi(w,\varepsilon) &= \frac{1}{\varepsilon}(I-\Pi)(P(v_0+\varepsilon w) + \varepsilon Q(v_0+\varepsilon w,\varepsilon)) = \\ &= \frac{1}{\varepsilon}(I-\Pi)\left(P(v_0) + \varepsilon P'\left(v_0+\overline{\varepsilon}(w,\varepsilon)w\right)w + \varepsilon Q(v_0+\varepsilon w,\varepsilon)\right),\end{aligned}$$

where $\widehat{\varepsilon}(w,\varepsilon), \widetilde{\varepsilon}(w,\varepsilon), \overline{\varepsilon}(w,\varepsilon) \in [0,\varepsilon]$. Using the fact that $P(v_0) = \Pi P'(v_0) = \Pi Q(v_0,0) = 0$ we get

$$\begin{aligned}\Psi(w,\varepsilon) \;&=\; \frac{1}{2}\Pi P''\left(v_0 + \widehat{\varepsilon}(w,\varepsilon)w\right) ww + \Pi Q'_v\left(v_0 + \widetilde{\varepsilon}(w,\varepsilon)w, \widetilde{\varepsilon}(w,\varepsilon)\right) w + \\ &+\Pi Q'_\varepsilon\left(v_0 + \widetilde{\varepsilon}(w,\varepsilon)w, \widetilde{\varepsilon}(w,\varepsilon)\right) + (I-\Pi)P'\left(v_0 + \overline{\varepsilon}(w,\varepsilon)w\right) w + (I-\Pi)Q(v_0+\varepsilon w,\varepsilon).\end{aligned}$$

From this formula we conclude that $\Psi \in C^0(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$.

Let us now prove that $\Psi'_w \in C^0(\mathbb{R}^n \times \mathbb{R}, \mathcal{L}(\mathbb{R}^n))$. The Taylor expansion applied to $P'(v_0+\varepsilon w)$ permits to write

$$\begin{aligned}\Pi\Psi'_w(w,\varepsilon) \;&=\; \frac{1}{\varepsilon^2}\Pi(\varepsilon P'(v_0+\varepsilon w) + \varepsilon^2 Q'_v(v_0+\varepsilon w,\varepsilon)) = \\ &=\; \frac{1}{\varepsilon^2}\Pi\left(\varepsilon P'(v_0) + \varepsilon^2 P''(v_0+\widetilde{\varepsilon}(w,\varepsilon)w)w + \varepsilon^2 Q'_v(v_0+\varepsilon w,\varepsilon)\right), \\ (I-\Pi)\Psi'_w(w,\varepsilon) \;&=\; \frac{1}{\varepsilon}(I-\Pi)\left(\varepsilon P'(v_0+\varepsilon w) + \varepsilon^2 Q'_v(v_0+\varepsilon w,\varepsilon)\right),\end{aligned}$$

where $\widetilde{\varepsilon}(w,\varepsilon) \in [0,\varepsilon]$. Taking into account that $\Pi P'(v_0) = 0$ we have

$$\Psi'_w(w,\varepsilon) = \Pi P''(v_0+\widetilde{\varepsilon}(w,\varepsilon)w)w + \Pi Q'_v(v_0+\varepsilon w,\varepsilon) + (I-\Pi)P'(v_0+\varepsilon w) + \varepsilon(I-\Pi)Q'_v(v_0+\varepsilon w,\varepsilon)$$

and so $\Psi'_w(w,\varepsilon) \to \Psi'_w(w_0,0)$ as $w \to w_0$ and $\varepsilon \to 0$. This concludes the proof. $\square$

Remark 1 *An example of linear projector which is invariant with respect to $P'(v_0)$ is the Riesz projector $\Pi_R : \mathbb{R}^n \to \mathbb{R}^n$ given by*

$$\Pi_R := \frac{1}{2\pi i}\int_\Gamma (\lambda I - P'(v_0))^{-1}\, d\lambda,$$

where Γ is a circumference centered at 0 and containing in its interior the only zero eigenvalue of $P'(v_0)$. In fact, by the Riesz decomposition theorem the subspaces $Im\,\Pi_R$ and $Ker\,\Pi_R$ are invariant with respect to $P'(v_0)$, $Im\,\Pi_R \bigoplus Ker\,\Pi_R = \mathbb{R}^n$ and $\Pi_R P'(v_0) = 0$.

We can now prove the following.

Theorem 1 *Assume that $P \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and $Q \in C^1(\mathbb{R}^n \times [0,1], \mathbb{R}^n)$. Let $v_0 \in \mathbb{R}^n$ be such that $P(v_0) = 0$ and $P'(v_0)$ is singular. Let $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ be a linear projector (not necessary one-dimensional) invariant with respect to $P'(v_0)$ with $P'(v_0)$ invertible on $(I-\Pi)\mathbb{R}^n$. Finally, assume that $\Pi Q(v_0,0) = 0$, $\Pi P''(v_0)\Pi r\, \Pi s = 0$ for any $r, s \in \mathbb{R}^n$, and that*

$$-\,\Pi P''(v_0)(I-\Pi)\left(P'(v_0)|_{(I-\Pi)\mathbb{R}^n}\right)^{-1} Q(v_0,0) + \Pi Q'_v(v_0,0) \tag{2.5}$$

is invertible on $\Pi\mathbb{R}^n$. Then there exists a unique $w_0 \in \mathbb{R}^n$ such that $\Psi(w_0,0) = 0$ and $\Psi'_w(w_0,0)$ is non-singular.

Proof. We start by showing the existence of a $w_0 \in \mathbb{R}^n$ such that $\Psi(w_0,0)=0$. First, observe that applying $(I-\Pi)$ to (2.3) we obtain the map $w \to (I-\Pi)P'(v_0)w + (I-\Pi)Q(v_0,0)$ and the equation

$$(I-\Pi)P'(v_0)w + (I-\Pi)Q(v_0,0) = (I-\Pi)P'(v_0)(I-\Pi)w + (I-\Pi)Q(v_0,0) = 0 \qquad (2.6)$$

is solvable with respect to $(I-\Pi)w$; in fact by our assumptions

$$w_1 = -\left(P'(v_0)\big|_{(I-\Pi)\mathbb{R}^n}\right)^{-1} Q(v_0,0).$$

is the solution of (2.6) with $w_1 \in (I-\Pi)\mathbb{R}^n$. Now, we solve the equation

$$\frac{1}{2}\Pi P''(v_0)(\Pi w + w_1)(\Pi w + w_1) + \Pi Q'_v(v_0,0)(\Pi w + w_1) + \Pi Q'_\varepsilon(v_0,0) = 0 \qquad (2.7)$$

with respect to Πw. By assumption $\Pi P''(v_0)\Pi r\, \Pi s = 0$ for any $r,s \in \mathbb{R}^n$, moreover $P''(v_0)ab = P''(v_0)ba$, hence we can rewrite equation (2.7) as follows

$$\Pi P''(v_0)w_1\Pi w + \Pi Q'_v(v_0,0)\Pi w = -\frac{1}{2}\Pi P''(v_0)w_1\, w_1 - \Pi Q'_v(v_0,0))w_1 - \Pi Q'_\varepsilon(v_0,0).$$

Since by assumption the operator $\Pi P''(v_0)w_1 + \Pi Q'(v_0,0)$ is invertible, the last equation has a unique solution w_2 with $w_2 \in \Pi\,\mathbb{R}^n$. Hence $w_0 = w_2 + w_1$ is a zero of $\Psi(w,0)$.
From Lemma 1 we have that Ψ is continuous at $(w_0,0)$, Ψ'_w exists and is continuous at $(w_0,0)$. To apply the classical implicit function theorem it remains to show that $\Psi'_w(w_0,0)$ is non-singular. We argue by contradiction assuming that there exists $h \neq 0$ such that

$$\Psi'_w(w_0,0)h = \Pi P''(v_0)w_0h + \Pi Q'_v(v_0,0)h + (I-\Pi)P'(v_0)h = 0. \qquad (2.8)$$

Applying $(I-\Pi)$ to (2.8) we obtain $(I-\Pi)P'(v_0)h = 0$ that is $(I-\Pi)h = 0$ and so $h = \Pi h$. Therefore,

$$\begin{aligned}\Pi P''(v_0)w_0h &= \Pi P''(v_0)\Pi w_0\Pi h + \Pi P''(v_0)(I-\Pi)w_0\Pi h = \\ &= -\Pi P''(v_0)\left(P'(v_0)|_{(I-\Pi)\mathbb{R}^n}\right)^{-1} Q(v_0,0)\Pi h\end{aligned}$$

and applying Π to (2.8) we obtain

$$-\Pi P''(v_0)\left(P'(v_0)\big|_{(I-\Pi)\mathbb{R}^n}\right)^{-1} Q(v_0,0)\Pi h + \Pi Q'_v(v_0,0)\Pi h = 0.$$

This contradicts our assumption and the proof is completed. $\square$

Remark 2 *The conclusions of Theorem 1 permit to apply the classical implicit function theorem to obtain the existence of a $\delta > 0$ such that the equation $\Psi(w,\varepsilon) = 0$ has, for any $\varepsilon \in [0,\delta]$, a unique solution w_ε such that $\|w_0 - w_\varepsilon\| \leq \delta$. Therefore, for $\varepsilon > 0$ small, there exists a family $\{w_\varepsilon\}$ of zeros of the map Ψ such that $w_\varepsilon \to w_0$ as $\varepsilon \to 0$.*

Moreover, under our regularity assumptions $\varepsilon \to \Phi'_v(v_0 + \varepsilon w_\varepsilon, \varepsilon)$ is a continuous map; thus, for any $\varepsilon > 0$ sufficiently small, there exists an eigenvalue λ_ε of $\Phi'_v(v_0 + \varepsilon w_\varepsilon, \varepsilon)$ with the property that $\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$.

We are now in the position to formulate the following result.

Theorem 2 *Assume all the conditions of Theorem 1 and that zero is a simple eigenvalue of $P(v_0)$. Let $v_0 = x(\theta_0)$, where $\theta \to x(\theta)$ is a C^2-parametrized curve of zeros of the map P. Let $\{w_\varepsilon\}$ and $\{\lambda_\varepsilon\}$ as in Remark 2. Let $\lambda_* \in \mathbb{R}$ be the eigenvalue of the operator $\Pi P''(v_0)w_0|_{\Pi\mathbb{R}^n} + \Pi Q'_v(v_0, 0)|_{\Pi\mathbb{R}^n}$. Then*

$$\lambda_\varepsilon = \varepsilon\lambda_* + o(\varepsilon).$$

Proof. Let l_ε be the unitary eigenvector of $\Phi'_v(v_0 + \varepsilon w_0, \varepsilon)$ associated to the eigenvalue λ_ε, namely

$$\Phi'_v(v_0 + \varepsilon w_\varepsilon, \varepsilon)l_\varepsilon = \lambda_\varepsilon l_\varepsilon. \tag{2.9}$$

Clearly,

$$l_\varepsilon \to \frac{\dot{x}_0(\theta_0)}{\|\dot{x}_0(\theta_0)\|} \quad \text{as} \quad \varepsilon \to 0. \tag{2.10}$$

Now we observe that

$$\Psi'_w(w, \varepsilon) = \frac{1}{\varepsilon}\left(\varepsilon\Phi'_v(v_0 + \varepsilon w, \varepsilon) - \varepsilon\Pi\Phi'_v(v_0 + \varepsilon w, \varepsilon) + \Pi\Phi'_v(v_0 + \varepsilon w, \varepsilon)\right)$$

and using (2.9) we get

$$\Pi\Psi'_w(w_\varepsilon, \varepsilon)l_\varepsilon = \frac{1}{\varepsilon}\Pi\Phi'_v(v_0 + \varepsilon w_\varepsilon, \varepsilon)l_\varepsilon = \frac{1}{\varepsilon}\lambda_\varepsilon\Pi l_\varepsilon \tag{2.11}$$

for any $\varepsilon > 0$ sufficiently small. By Lemma 1 as $\varepsilon \to 0$ we have

$$\Pi\Psi'_w(w_\varepsilon, \varepsilon)l_\varepsilon \quad \to \quad \Pi P''(v_0)w_0\frac{\dot{x}_0(\theta_0)}{\|\dot{x}_0(\theta_0)\|} + \Pi Q'_v(v_0, 0)\frac{\dot{x}_0(\theta_0)}{\|\dot{x}_0(\theta_0)\|}.$$

From this, by (2.11) we have that $\dfrac{\lambda_\varepsilon}{\varepsilon} \to a \in \mathbb{R}$ as $\varepsilon \to 0$ and

$$\Pi P''(v_0)w_0\dot{x}_0(\theta_0) + \Pi Q'_v(v_0, 0)\dot{x}_0(\theta_0) = a\dot{x}_0(\theta_0).$$

Therefore, $a = \lambda_*$, and the proof is completed. □

3 An application to periodically perturbed autonomous equations

In this Section we show that the results of the previous Section can be straight apply to the problem of bifurcation of asymptotically stable T-periodic solutions to T-periodically perturbed

autonomous systems. Specifically, by showing that our function (2.5) is nothing else than the Malkin's bifurcation function, as far as periodically perturbed autonomous systems are concerned, we prove the existence of a unique branch of asymptotically stable periodic solutions emanating from the family of periodic solutions represented by limit cycle x_0 of the unperturbed system.

The system under consideration is the following

$$\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon). \tag{3.1}$$

where $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, $g \in C^1(\mathbb{R} \times \mathbb{R}^n \times [0,1], \mathbb{R}^n)$ is T-periodic and $\varepsilon > 0$ is the bifurcation parameter. We assume that the unique solution of any Cauchy problem associated to (3.1) is defined on $[0, T]$.

We associate to the unperturbed autonomous system

$$\dot{x} = f(x) \tag{3.2}$$

the Malkin's bifurcation function [11]

$$M(\theta) = \int_0^T \langle g(t, x_0(t+\theta), 0), z_0(t+\theta) \rangle \, dt$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^n$ and z_0 is the T-periodic function of the adjoint system

$$\dot{z} = -(f'(x_0(t)))^* z$$

of the linearized system of

$$\dot{y} = f'(x_0(t))y$$

of autonomous system (3.2). Let $\theta \in [0, T]$, we define the projector $\Pi : \mathbb{R}^n \to \mathbb{R}^n$ as follows

$$\Pi\xi = \dot{x}_0(\theta) \langle \xi, z_0(\theta) \rangle .$$

Finally, we convert the problem of finding T-periodic solutions to (3.1) into the fixed point problem for the associated Poincaré map $\mathcal{P}_\varepsilon$ as illustrated in the following. We consider the function $x : [0, T] \times \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ given by

$$x(t, v, \varepsilon) = x(t)$$

for all $t \in [0, T]$, where $x(t)$ is the solution of systems equation (3.1). The Poincaré map for system (3.1) is defined by

$$\mathcal{P}_\varepsilon(v) = x(T, v, \varepsilon).$$

The functions P and Q of the previous section are defined as $P(v) = \mathcal{P}_0(v) - v$, $Q(v, \varepsilon) = \dfrac{\mathcal{P}_\varepsilon(v) - \mathcal{P}_0(v)}{\varepsilon}$ that leads to

$$\mathcal{P}_\varepsilon(v) - v = P(v) + \varepsilon Q(v, \varepsilon).$$

Observe that, since $P(x_0(\theta)) = 0$ for any θ, we have that $P'(x_0(\theta))\,\dot{x}_0(\theta) = 0$ and so

$$(\mathcal{P}_0)'(x_0(\theta)) - I = P'(x_0(\theta))$$

is a singular $n \times n$ matrix for any $\theta \in [0, T]$.

With x_0, z_0, Π, P, Q as introduced before we have the following two results. The first one provides a representation formula for the Malkin's bifurcation function, the second one a formula for its derivative.

Lemma 2 *For any $\theta \in [0, T]$ the limit $Q(v, 0) := \lim_{\varepsilon \to 0} Q(v, \varepsilon)$ exists and*

$$M(\theta) = \langle Q(x_0(\theta), 0), z_0(\theta) \rangle \, .$$

Moreover, $Q \in C^1(\mathbb{R}^n \times [0, 1], \mathbb{R}^n)$.

Proof. Differentiating with respect to time one can see that the function $y(t) = \dfrac{\partial}{\partial \varepsilon} x(t, x_0(\theta), \varepsilon)$ evaluated at $\varepsilon = 0$ solves, for any $\theta \in [0, T]$, the Cauchy problem

$$\dot{y} = f'(x_0(t + \theta))y + g(t, x_0(t + \theta), 0), \quad y(0) = 0.$$

A direct computation shows that

$$\frac{d}{dt} \langle y(t), z_0(t + \theta) \rangle = \langle g(t, x_0(t + \theta), 0), z_0(t + \theta) \rangle$$

and, integrating over the period, yields

$$M(\theta) = \langle y(T), z_0(\theta) \rangle = \langle Q(x_0(\theta), 0), z_0(\theta) \rangle \, .$$

□

Lemma 3 *For any $\theta \in [0, T]$ we have*

$$M'(\theta) = \left\langle -P''(x_0(\theta))(I - \Pi) \left(P'(x_0(\theta))\big|_{(I - \Pi)\mathbb{R}^n} \right)^{-1} Q(x_0(\theta), 0)\dot{x}_0(\theta) + Q'_v(x_0(\theta), 0)\dot{x}_0(\theta), z_0(\theta) \right\rangle . \tag{3.3}$$

Proof. By Perron's Lemma [12] we have that

$$\langle \dot{x}(\theta), z_0(\theta) \rangle = \langle \dot{x}(0), z_0(0) \rangle$$

for any $\theta \in [0, T]$. Without loss of generality we may assume that $\langle \dot{x}(0), z_0(0) \rangle = 1$. As a consequence, by the definition of the projector Π, we get

$$\langle \xi, z_0(\theta) \rangle = \langle \Pi\, \xi, z_0(\theta) \rangle \, , \tag{3.4}$$

for any $\theta \in [0,T]$. Therefore

$$\left\langle P'(x_0(\theta))h, z_0(\theta)\right\rangle = \left\langle \Pi\, P'(x_0(\theta))(I-\Pi)h, z_0(\theta)\right\rangle = 0,$$

for any $\theta \in [0,T]$ and any $h \in \mathbb{R}^n$. Then, by deriving with respect to θ, we obtain

$$\left\langle P'(x_0(\theta))h, \dot{z}_0(\theta)\right\rangle = \left\langle -P''(x_0(\theta))\dot{x}_0(\theta)h, z_0(\theta)\right\rangle,$$

for any $\theta \in [0,T]$ and any $h \in \mathbb{R}^n$. Therefore, we can rewrite the left hand side of (3.3) with $(I-\Pi)\left(P'(x_0(\theta))|_{(I-\Pi)\mathbb{R}^n}\right)^{-1} Q(x_0(\theta),0) = h$ as follows

$$\left\langle P'(x_0(\theta))(I-\Pi)\left(P'(x_0(\theta))|_{(I-\Pi)\mathbb{R}^n}\right)^{-1} Q(x_0(\theta),0), \dot{z}_0(\theta)\right\rangle + \left\langle Q'_v(x_0(\theta),0)\dot{x}_0(\theta), z_0(\theta)\right\rangle$$

or equivalently,

$$\langle Q(x_0(\theta),0), \dot{z}_0(\theta)\rangle + \left\langle Q'_v(x_0(\theta),0)\dot{x}_0(\theta), z_0(\theta)\right\rangle,$$

which is the derivative of $M(\theta)$ at any $\theta \in [0,T]$ according to the formula given by Lemma 2. $\square$

Finally, we can prove the following.

Theorem 3 *Assume that there exists $\theta_0 \in [0,T]$ such that $(\mathcal{P}_0)'(x_0(\theta_0))$ has $n-1$ eigenvalues with negative real parts, $M(\theta_0) = 0$ and $M'(\theta_0) < 0$. Then, for $\varepsilon > 0$ sufficiently small, equation (3.1) has a unique T-periodic solution x_ε such that $x_\varepsilon(t) \to x_0(t+\theta_0)$ as $\varepsilon \to 0$ uniformly in $[0,T]$. Moreover the solutions $\{x_\varepsilon\}$ are asymptotically stable.*

Proof. Let $v_0 = x_0(\theta_0)$, from Lemma 2 we have

$$\Pi Q(x_0(v_0),0) = \dot{x}_0(\theta_0)\,\langle Q(v_0,0), z_0(\theta_0)\rangle = \dot{x}_0(\theta_0)M(\theta_0) = 0.$$

By (3.4) we obtain

$$M'(\theta_0) = \left\langle -\Pi P''(v_0)(I-\Pi)\left(P'(v_0)|_{(I-\Pi)\mathbb{R}^n}\right)^{-1} Q(v_0,0)\dot{x}_0(\theta_0) + \Pi Q'_v(v_0,0)\dot{x}_0(\theta_0), z_0(\theta_0)\right\rangle \neq 0,$$

and so (2.5) is invertible on $\Pi\mathbb{R}^n$. Moreover, from the fact that $P(x_0(\theta)) = 0$ for any $\theta \in [0,T]$, we obtain that

$$P''(v_0)\dot{x}_0(\theta_0)\dot{x}_0(\theta_0) + P'(v_0)x_0''(\theta_0) = 0$$

Since $\Pi P'(v_0)x_0''(\theta_0) = \Pi P'(v_0)\Pi x_0''(\theta_0) = 0$ we have that $\Pi P''(v_0)\,\Pi r\,\Pi s = 0$ for any $r, s \in \mathbb{R}^n$. Therefore, all the conditions of Theorem 1 are satisfied and so, compare Remark 2, equation (3.1) has a unique T-periodic solution x_ε satisfying

$$\left\| w_0 - \frac{x_\varepsilon(0) - v_0}{\varepsilon}\right\| \leq \delta,$$

with $\Psi(w_0,0)=0$. Moreover

$$\Pi P''(v_0)w_0\dot{x}_0(\theta_0)+\Pi Q'_v(v_0,0)\dot{x}_0(\theta_0)=\lambda_*\,\dot{x}_0(\theta_0).$$

But

$$\text{sign}\,\lambda_*=\text{sign}\,\left\langle \Pi P''(v_0)w_0\dot{x}_0(\theta_0)+\Pi Q'_v(v_0,0)\dot{x}_0(\theta_0),z_0(\theta_0)\right\rangle=\text{sign}\,M'(\theta_0)=-1$$

Therefore, from Theorem 2 there exists $\lambda_\varepsilon=\varepsilon\lambda_*+o(\varepsilon)$ eigenvalue of $(\mathcal{P}_\varepsilon)'(x_\varepsilon(0))-I$. This implies that

$$\det\left((\mathcal{P}_\varepsilon)'(x_\varepsilon(0))-I-\lambda_\varepsilon I\right)=0.$$

Hence, $\rho_\varepsilon=1+\lambda_\varepsilon=1+\lambda_*\varepsilon+o(\varepsilon)$ is an eigenvalue of $(\mathcal{P}_\varepsilon)'(x_\varepsilon(0))$ converging to 1 as $\varepsilon\to 0$. Since $\lambda_*<0$, then $|\rho_\varepsilon|<1$ for $\varepsilon>0$ sufficiently small. This ends the proof. □

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